On the Connection between Generalized Hypergeometric Functions and Dilogarithms

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Abstract

Several integrals involving powers and ordinary hypergeometric functions are rederived by means of a generalized hypergeometric function of two variables (Appell's function) recovering some well-known expressions as particular cases. Simple connections between dilogarithms and a kind of Appell's function are shown. A relationship is generalized to polylogarithms.

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Hypergeometric functions play an important role in mathematical physics since they are related to a wide class of special functions appearing in a large variety of fields. In particular, it is well-known a long time ago that integrals emerging from loop calculations in Feynman diagrams can be written in terms of hypergeometric functions [1]. More recently, generalized hypergeometric functions of one or several variables have been used in the evaluation of scalar one-loop Feynman integrals [2] or multiloop ones [3].

In this work we firstly rederive [1] an integral expression involving two ordinary Gauss' functions yielding a generalized hypergeometric function of two variables (Appell's function). Several formulae appearing in standard tables (*e.g.* Gradshteyn and Ryzhik [5]) of utility for the evaluation of Feynman loop integrals are obtained as particular cases. Moreover, we have shown a simple relationship between a kind of Appell's function and dilogarithms [6], contributing to enlarge the knowledge on the connection between them. In the appendices at the end of the paper we present a brief survey on the generalized Gauss' functions establishing the notation employed and revising some of their properties needed in this work.

expression 1

$$\int_0^1 du\ u^{\gamma-1}(1-u)^{\rho-1}\ {}_2F_1[\sigma,\eta;\gamma;zu]\ {}_2F_1[\alpha,\beta;\rho;k(1-u)] =$$

$$= \frac{\Gamma(\gamma)\Gamma(\rho)}{\Gamma(\gamma+\rho)}\ F_3[\alpha,\sigma,\beta,\eta;\gamma+\rho;k,z] \tag{1}$$

provided that $Re(\gamma)>0$, $Re(\rho)>0$, $|arg(1-k)|<\pi$, $|arg(1-z)|<\pi$.

Proof. We will first show that Eq. (1) holds in the domain of convergence of the series. Expanding one of the two ${}_2F_1$ functions as a power series leads to:

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\rho)_n}\frac{k^n}{n!}\int_0^1 du\ u^{\gamma-1}(1-u)^{\rho+n-1}\ {}_2F_1[\sigma,\eta;\gamma;zu] \quad ;\ |k|<1,\ |z|<1$$

where we have interchanged the order of summation and integration on account of the dominated convergence theorem of Lebesgue, provided that $Re(\gamma)>0$, $Re(\rho)>0$. Now, performing the integration over u one gets from (A.3):

$$\Gamma(\gamma)\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\rho)_n}\frac{k^n}{n!}\frac{\Gamma(\rho+n)}{\Gamma(\gamma+\rho+n)}\ {}_3F_2[\gamma,\sigma,\eta;\gamma+\rho+n,\gamma;z] =$$

$$= \Gamma(\gamma)\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\rho)_n}\frac{k^n}{n!}\frac{\Gamma(\rho+n)}{\Gamma(\gamma+\rho+n)}\ {}_2F_1[\sigma,\eta;\gamma+\rho+n;z]$$

where a cancellation between two parameters in the ${}_3F_2$ function occurred.

[1] An exhaustive set of integrals involving generalized Gauss functions containing ours as a particular case can be found in [4].

Finally, using that: $\Gamma(\rho+n)=\Gamma(\rho)(\rho)_n$, $\Gamma(\gamma+\rho+n)=\Gamma(\gamma+\rho)(\gamma+\rho)_n$, one arrives at

$$\frac{\Gamma(\gamma)\Gamma(\rho)}{\Gamma(\gamma+\rho)}\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma+\rho)_n}\frac{k^n}{n!}\ {}_2F_1[\sigma,\eta;\gamma+\rho+n;z]$$

which leads to Eq. (1) at once in virtue of (B.2). Moreover, the integral (1) furnishes a single-valued function of two variables beyond the domain of convergence of the series by imposing the cuts: $|arg(1-k)|<\pi$, $|arg(1-z)|<\pi$.

expression 1.1

Setting $\rho=\delta-\gamma$, $\alpha=\delta-\sigma$ and $\beta=\delta-\eta$ in Eq. (1) the formula 7.512.7 of reference [5] is recovered: [2]

$$\int_0^1 du\ u^{\gamma-1}(1-u)^{\delta-\gamma-1}\ {}_2F_1[\sigma,\eta;\gamma;zu]\ {}_2F_1[\delta-\sigma,\delta-\eta;\delta-\gamma;k(1-u)]=$$

$$=\frac{\Gamma(\gamma)\Gamma(\delta-\gamma)}{\Gamma(\delta)}\ (1-k)^{\sigma+\eta-\delta}\ {}_2F_1[\sigma,\eta;\delta;k+z-kz] \qquad (2)$$

provided that $Re(\delta)>Re(\gamma)>0$, $|arg(1-k)|<\pi$, $|arg(1-z)|<\pi$.

This can be directly obtained from Eq. (1) taking further into account the property (B.6) which here implies:

$$F_3[\delta-\sigma,\sigma,\delta-\eta,\eta;\delta;k,z]=(1-k)^{\sigma+\eta-\delta}\ {}_2F_1[\sigma,\eta;\delta;k+z-kz]$$

expression 1.2

With the aid of (A.4) the left hand side of Eq. (1) can be written as:

$$\int_0^1 du\ u^{\gamma-1}(1-u)^{\rho-1}(1-k(1-u))^{-\alpha}\ {}_2F_1[\sigma,\eta;\gamma;zu]\ {}_2F_1\left[\alpha,\rho-\beta;\rho;\frac{k(1-u)}{(k-1-ku)}\right]$$

Now, let us assume that z and k are related through $k=z/(z-1)$. Then

$$(1-z)^{\alpha}\int_0^1 du\ u^{\gamma-1}(1-u)^{\rho-1}(1-zu)^{-\alpha}\ {}_2F_1[\sigma,\eta;\gamma;zu]\ {}_2F_1[\alpha,\rho-\beta;\rho;\frac{z(1-u)}{1-zu}]=$$

$$=\frac{\Gamma(\gamma)\Gamma(\rho)}{\Gamma(\gamma+\rho)}\ F_3[\alpha,\sigma,\beta,\eta;\gamma+\rho;z/(z-1),z]$$

Next, let us suppose further that $\beta=\gamma+\rho-\eta$. Then taking into account consecutively the properties (B.5) and (B.4) the right hand side of the last expression becomes:

$$\frac{\Gamma(\gamma)\Gamma(\rho)}{\Gamma(\gamma+\rho)}\ (1-z)^{\alpha}F_1[\eta;\sigma;\alpha;\gamma+\rho;z,z]=\frac{\Gamma(\gamma)\Gamma(\rho)}{\Gamma(\gamma+\rho)}\ (1-z)^{\alpha}\ {}_2F_1[\sigma+\alpha,\eta;\gamma+\rho;z]$$

[2]Except the exponent of $(1-k)$ which in our notation would read: $2\sigma-\delta$. Clearly this is an error since the result should be invariant under the interchange of σ and η, as the l.h.s. certainly is. The original source [7] is equally wrong.

Hence one recovers the formula 7.512.8 of ref. [5]:

$$\int_0^1 du\ u^{\gamma-1}(1-u)^{\rho-1}(1-zu)^{-\alpha}\ {}_2F_1[\sigma,\eta;\gamma;zu]\ {}_2F_1\left[\alpha,\eta-\gamma;\rho;\frac{z(1-u)}{(1-zu)}\right] =$$

$$= \frac{\Gamma(\gamma)\Gamma(\rho)}{\Gamma(\gamma+\rho)}\ {}_2F_1[\sigma+\alpha,\eta;\gamma+\rho;z] \tag{3}$$

provided that $Re(\gamma) > 0$, $Re(\rho) > 0$, $|arg(1-z)| < \pi$.

Let us now go back again to Eq. (1) and consider $\eta = \gamma$ as a new special case. Then two parameters of a hypergeometric function in the integrand cancel, i.e. ${}_2F_1[\sigma,\gamma;\gamma;zu] = {}_1F_0[\sigma;zu] = (1-zu)^{-\sigma}$, yielding:

expression 2

$$\int_0^1 du\ u^{\gamma-1}(1-u)^{\rho-1}(1-zu)^{-\sigma}\ {}_2F_1[\alpha,\beta;\gamma;ku] =$$

$$= \frac{\Gamma(\gamma)\Gamma(\rho)}{\Gamma(\gamma+\rho)}\ (1-z)^{-\sigma} F_3[\alpha,\sigma,\beta,\rho;\gamma+\rho;k,z/(z-1)] \tag{4}$$

provided that $Re(\gamma) > 0$, $Re(\rho) > 0$, $|arg(1-k)| < \pi$, $|arg(1-z)| < \pi$.

Proof. It follows inmediately as a particular case of the expression **1** by means of the change of the integration variable: $u\to 1-u$ and interchanging the γ and ρ parameters.

An alternative (direct) proof is achieved with the aid of the integral representation of the F_3 Appell's function. Starting from (B.3) and making the consecutive changes of the integration variables: $v\to 1-v$ and $u\to uv$ it follows that

$$\frac{\Gamma(\beta)\Gamma(\rho)\Gamma(\gamma-\beta)}{\Gamma(\gamma+\rho)}\ F_3[\alpha,\sigma,\beta,\rho;\gamma+\rho;k,z/(z-1)]\ =$$

$$(1-z)^{\sigma}\int_0^1\int_0^1 dv\ du\ v^{\gamma-1}u^{\beta-1}(1-v)^{\rho-1}(1-u)^{\gamma-\beta-1}(1-zv)^{-\sigma}(1-kvu)^{-\alpha}$$

Hence the expression **2** is immediately obtained by expressing ${}_2F_1[\alpha,\beta;\gamma;kv]$ in its Euler's integral representation (A.2).

expression 2.1

Setting $k = 1$ in expression **2** reproduces the result 7.512.9 of ref. [5]:

$$\int_0^1 du\ u^{\gamma-1}(1-u)^{\rho-1}(1-zu)^{-\sigma}\ {}_2F_1[\alpha,\beta;\gamma;u] =$$

$$= \frac{\Gamma(\gamma)\Gamma(\rho)\Gamma(\gamma+\rho-\alpha-\beta)}{\Gamma(\gamma+\rho-\alpha)\Gamma(\gamma+\rho-\beta)}(1-z)^{-\sigma}\ {}_3F_2[\rho,\sigma,\gamma+\rho-\alpha-\beta;\gamma+\rho-\alpha,\gamma+\rho-\beta;z/(z-1)]$$

provided additionally that: $Re(\gamma+\rho-\alpha-\beta) > 0$.

This can be easily shown by rewriting the power expansion of F_3 following (B.2) in terms of ${}_2F_1[\alpha, \beta; \gamma+\rho+n; 1]$ supposed the convergence of the series, and using the Gauss' summation relation:

$$ {}_2F_1[a, b; c; 1] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \qquad ; \; Re(c-a-b) > 0 $$

Particular values of the parameters of F_3

Let us now take the particular values of the parameters: $\alpha = \beta = \sigma = \eta = \gamma = 1$ and $\rho = 2$ in the $F_3[\alpha, \sigma, \beta, \eta; \gamma + \rho; x, y]$ Appell's function. From our expression **1**, it is easy to see that making $y = 1$ ones gets

$$ x \; F_3[1, 1, 1, 1; 3; x, y = 1] \; = \; 2x \; {}_3F_2[1, 1, 1; 2, 2; x] \; = \; 2 \; Li_2(x) \qquad (5) $$

where Eq. (A.8) has been taken into account. In fact, one can get the same result by expanding $F_3[1, 1, 1, 1; 3; x, 1]$ in terms of ${}_2F_1[1, 1; 3+m; 1]$ and using the Gauss' summation relation. (See also appendix B for a generalization to polylogarithms.) The restriction $|x| < 1$ can be dropped on account of analytic continuation, extending the domain of analyticity over the complex x-plane cut from 1 to ∞ along the real axis. It is obvious from symmetry, that an equivalent expression for y must be satisfied. In fact, Eq. (5) can be viewed as a particular case of a more general relationship between this Appell's series and dilogarithms:

expression 3

$$ \frac{1}{2} xy \; F_3[1, 1, 1, 1; 3; x, y] = Li_2(x) + Li_2(y) - Li_2(x + y - xy) \qquad (6) $$

$|arg(1-x)| < \pi$, $|arg(1-y)| < \pi$.

Proof. This formula can be again derived from expression **1** by calculating directly the integral in terms of dilogarithms. Instead, we will prove it by differentiating both sides with respect to x and y consecutively. Expanding the Appell's function as a double series, the result of differentiating the l.h.s. reads:

$$ \frac{1}{2} \; F_3[1, 2, 2, 1; 3; x, y] \qquad (7) $$

Now, invoking the property (B.6):

$$ F_3[\alpha, \gamma - \alpha, \beta, \gamma - \beta; \gamma; x, y] = (1-y)^{\alpha+\beta-\gamma} \; {}_2F_1[\alpha, \beta; \gamma; x + y - xy] $$

which is valid in a suitable small open polydisc centered at the origin, we conclude that (7) can be rewritten as

$$ \frac{1}{2} \; {}_2F_1[1, 2; 3; x + y - xy] $$

Next, differentiating twice the r.h.s of Eq. (6) one gets:

$$-\frac{1}{x+y-xy}[1+\frac{1}{x+y-xy}\ln(1-(x+y-xy))] \;=\; \frac{1}{2}\,{}_2F_1[1,2;3;x+y-xy]$$

the last step coming from (A.5). Then both sides in Eq. (6) would differ in $f(x)+g(y)$:

$$\frac{1}{2}xy\;F_3[1,1,1,1;3;x,y] = Li_2(x)+Li_2(y)-Li_2(x+y-xy)+f(x)+g(y)$$

where $f(x)$ and $g(y)$ are functions to be determined by taking particular values of the variables. Setting $x=0$ and $y=0$ it is easy to see that $f(x)=g(y)\equiv 0$.

Now, by analytic continuation we dispense with the restriction on the small polydisc, extending its validity to a suitable domain of C^2: in order to get a single-valued function, with a well-defined branch for each dilogarithm in Eq. (6), we assume further that $|arg(1-x)|<\pi$, $|arg(1-y)|<\pi$. [3]

expression 3.1

$$x^2\;F_3[1,1,1,1;3;x,-x] \;=\; Li_2(x^2) \tag{8}$$

for x real.

Proof. It follows directly from Eq. (6) using the relation $Li_2(x)+Li_2(-x)=\frac{1}{2}Li_2(x^2)$.

expression 3.2

$$x^2\;F_3[1,1,1,1;3;x,x] \;=\; 4\;Li_2\left(\frac{1}{2-x}\right) \;+\; 2\ln^2(2-x) \;-\; \frac{\pi^2}{3} \tag{9}$$

for x real and less than unity.

Proof. It follows directly from Eq. (6) using the relation: $Li_2(2x-x^2)=2Li_2(x)-2Li_2(1/(2-x))+\pi^2/6-\ln^2(2-x)$ [6].

expression 3.3

$$lim_{y\to 0}\;\frac{xy}{2}\;F_3[1,1,1,1;3;x,y] \;=\; y\;\left[\;1\;+\;\frac{1-x}{x}\ln(1-x)\;\right] \tag{10}$$

Proof. It follows directly from Eq. (6) using the relation: ${}_2F_1[1,1;3;x]=(1-x)^{-1}\;{}_2F_1[1,2;3;x/(x-1)]$ and (A.5). If besides $x\to 0$, the limit $xy/2$ is quickly recovered.

The set of expressions **3** provide new connections (not shown in literature to our knowledge) between dilogarithms and a certain F_3 Appell's function.

[3] Observe that then each function of one complex variable obtained from (6) by fixing the other variable is analytic in the corresponding subset of C^2. Then the function of two variables $\frac{1}{2}xyF_3$ is analytic according to the theorem of Hartogs-Osgood

Appendices

A

Generalized Gauss' Functions of one variable

Hypergeometric functions can be introduced at first as series within a certain domain of convergence [8] [9] [10]. We write, using the abbreviate notation:

$$ {}_pF_q[\{a\}_p; \{b\}_q; z] \ = \ \sum_{n=0}^{\infty} \frac{(a_1)_n ... (a_p)_n}{(b_1)_n ... (b_q)_n} \frac{z^n}{n!} \tag{A.1} $$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ stands for the Pochhammer symbol. We suppose that none of the denominator parameters is a negative integer or zero. This series converges for all values of z, real or complex, when $p \leq q$, and for $|z| < 1$ when $p = q+1$. In the latter case, it also converges (absolutely) on the circle $|z| = 1$ if $Re\ (\sum_{i=1}^{q} b_i - \sum_{i=1}^{p} a_i) > 0$. If $p > q+1$, the series never converges, except either when $z = 0$ or when the series terminates, that is when one at least of the a parameters is zero or a negative integer.

Hypergeometric series admit in general an integral representation of the Euler's type [10] [11] [12] , which permits the corresponding analytic continuation in the complex z-plane beyond the unit disc.

For the ordinary hypergeometric series, we have:

$$ {}_2F_1[a, b; c; z] \ = \ \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 du\ u^{b-1}(1-u)^{c-b-1}(1-zu)^{-a} \tag{A.2} $$

with $Re(c) > Re(b) > 0$. In order to get a single-valued analytic function in the whole complex z-plane we will follow the customary convention of assuming a cut along the real axis from 1 to ∞.

For the generalized hypergeometric function of one variable the integral representation of the Euler's type is:

$$ {}_pF_q[a_1, \{a\}_{p-1}; b_1, \{b\}_{q-1}; z] \ = \ \frac{\Gamma(b_1)}{\Gamma(a_1)\Gamma(b_1-a_1)} \int_0^1 du\ u^{a_1-1}(1-u)^{b_1-a_1-1}\ {}_{p-1}F_{q-1}[\{a\}_{p-1}; \{b\}_{q-1}; zu] \tag{A.3} $$

under the constraints: $p \leq q+1$, $Re(b_1) > Re(a_1) > 0$ and none of $b_i,\ i = 1...q$, is zero or a negative integer, giving the analytic continuation in the whole complex z-plane, cut along the positive axis from 1 to ∞ again.

A well-known transformation between Gauss' hypergeometric functions of one variable, needed in the main text is: [12]

$$ {}_2F_1[a, b; c; z] = (1-z)^{c-b-a}\ {}_2F_1[c-a, c-b; c; z] $$
$$ = \ (1-z)^{-a}\ {}_2F_1[a, c-b; c; z/(z-1)] \ = \ (1-z)^{-b}\ {}_2F_1[c-a, b; c; z/(z-1)] \tag{A.4} $$

An interesting relation between an ordinary Gauss' function and an elementary function not usually shown in specialized tables is:

$$ z\ {}_2F_1[1, 2; 3; z] = -2\ [\ 1 + \frac{1}{z} \ln{(1-z)}\ ] \tag{A.5} $$

which can be proved by expanding both sides as power series.

The dilogarithm and its relation to the ${}_3F_2$ function

The dilogarithmic function is defined as: [13] [6]

$$Li_2(z) = -\int_0^1 du \, \frac{\ln(1 - zu)}{u} \tag{A.6}$$

for values of z real or complex. If $|z| < 1$, the dilogarithm may be expanded as the power series:

$$Li_2(z) = \sum_{n=0}^{\infty} \frac{z^{n+1}}{(n+1)^2} \tag{A.7}$$

corresponding to the principal value. We can also write:

$$Li_2(z) = z \int_0^1 du \; {}_2F_1[1, 1; 2; zu] = z \; {}_3F_2[1, 1, 1; 2, 2; z] \tag{A.8}$$

The derivative of the dilogarithm is

$$\frac{d}{dz} Li_2(z) = -\frac{\ln(1 - z)}{z} \tag{A.9}$$

B

Generalized Gauss' Functions of two variables : Appell's functions

In this paper we are involved in particular with the F_3 Appell's function [5] [9] [10] [11], so we write its series expansion:

$$F_3[a,a',b,b';c;x,y] \;=\; \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_m(a')_n(b)_m(b')_n}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!} \tag{B.1}$$

which exists for all real or complex values of a, a', b, b', and c except c a negative integer. With regard to its convergence, the F_3 series is absolutely convergent when both $|x| < 1$ and $|y| < 1$. Then there is no problem with internal rearrangements of the series.

The F_3 function can be rewritten in terms of ordinary Gauss' functions:

$$F_3[a,a',b,b';c;x,y] \;=\; \sum_{m=0}^{\infty}\frac{(a)_m(b)_m}{(c)_m}\frac{x^m}{m!}\;{}_2F_1[a',b';c+m;y] \tag{B.2}$$

where we have made use of the relation: $(c)_{m+n} = (c)_m(c+m)_n$.

Moreover, the F_3 function admits the following integral representation: [5] [10]

$$F_3[a,a',b,b';c;x,y] = \frac{\Gamma(c)}{\Gamma(b)\Gamma(b')\Gamma(c-b-b')}\times \tag{B.3}$$

$$\int\!\!\int du\ dv\ u^{b-1}v^{b'-1}(1-u-v)^{c-b-b'-1}(1-xu)^{-a}(1-yv)^{-a'}$$

where the integral is taken over the triangular region $0{\le}u$, $0{\le}v$, $u+v{\le}1$, under the conditions: $Re(b) > 0$, $Re(b') > 0$, $Re(c-b-b') > 0$. This expression furnishes a single-valued analytic function in the domain defined by the Cartesian product of the complex planes of x and y with the restrictions $|arg(1-x)| < \pi$, $|arg(1-y)| < \pi$. Hence, the order of integration may be reversed according to Fubini's theorem.

Some properties and relations between Appell's functions needed in this paper are given below: [5] [10]

$$F_1[a;b,b';c;x,x] \;=\; {}_2F_1[a,b+b';c;x] \;=\; {}_2F_1[b+b',a;c;x] \tag{B.4}$$

$$F_3[a,c-a,b,b';c;x,y/(y-1)] = F_3[b',b,c-a,a;c;y/(y-1),x] = (1-y)^{b'}F_1[a;b,b';c;x,y] \tag{B.5}$$

$$F_3[a,c-a,b,c-b;c;x,y] = F_3[c-a,a,c-b,b;c;y,x] = (1-y)^{a+b-c}\;{}_2F_1[a,b;c;x+y-xy] \tag{B.6}$$

The polylogarithm and its relation to the generalized Campé de Fériet function $F_B^{(2)}$

The polylogarithm $Li_q(z)$ is defined as a series as

$$Li_q(z) \;=\; \sum_{n=0}^{\infty} \frac{z^{n+1}}{(n+1)^q} \qquad (q>1) \tag{B.7}$$

which can be expressed according to

$$Li_q(z) = z \int_0^1 \; du \;{}_qF_{q-1}[\{1\}_q; \{2\}_{q-1}; zu] \;=\; z\;{}_{q+1}F_q[\{1\}_{q+1}; \{2\}_q; z]$$

A generalized Campé de Fériet function [4], of particular interest for us, is defined as

$$F_B^{(2)}[\{b\}_r, \{b'\}_s; \{d\}_t; c; x, y] \;=\; \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(b_1)_m ... (b_r)_m (b'_1)_n ... (b'_s)_n}{(d_1)_m ... (d_t)_m \; (c)_{m+n}} \; \frac{x^m}{m!}\frac{y^n}{n!} \tag{B.8}$$

Thus the following equality is satisfied

$$x \; F_B^{(2)}[\{1\}_q, \{1\}_2; \{2\}_{q-2}; 3; x, y=1] \;=\; 2 \; Li_q(x) \tag{B.9}$$

which is the generalization of Eq. (5).

[4]Campé de Fériet functions are special cases of generalized Lauricella functions of two variables [2]

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